Sampled-Data Control Design
A Differential LMI Approach

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Outline

Notation

- Real ($\mathbb{R}$), nonnegative real ($\mathbb{R}_+$) and natural numbers ($\mathbb{N}$)
- The **trace** function is $\mathbf{tr}(\cdot)$
- Positive definite symmetric real matrix $X > 0$
- Maximum eigenvalue $\sigma_{max}(\cdot)$
- Norm bounded signals - continuous $\mathcal{L}_2$ and discrete-time ℓ_2
- $\psi(t^-) = \lim_{\varepsilon \to 0^-} \psi(t + \varepsilon)$
- $f(t_k) = f(t)|_{t=t_k}$ for $k \in \mathbb{N}$ is denoted $f[k]$

Motivation

- **Stability:** Matrix $A \in \mathbb{R}^{n \times n}$ is
  - Hurwitz stable $\quad \mathrm{Re}\{\lambda_i(A)\} < 0,\ \forall i = 1, \cdots, n$
  - Schur stable $\quad |\lambda_i(A)| < 1,\ \forall i = 1, \cdots, n$

$$\Downarrow$$

Simple and useful relationship: For any $h > 0$

$$A \text{ Hurwitz stable} \iff e^{Ah} \text{ Schur stable}$$

Motivation

- Consider $h > 0$ and a polytopic convex set $\mathcal{A}_c \subset \mathbb{R}^{n \times n}$

$$\underbrace{\mathcal{A}_d}_{\textit{generic!}} = \left\{ e^{Ah} \;:\; A \in \underbrace{\mathcal{A}_c}_{\textit{polytopic}} \right\}$$

- **Robust stability (analysis):**

$$\underbrace{\forall A \in \mathcal{A}_c}_{\textit{Hurwitz}} \iff \underbrace{\forall e^{Ah} \in \mathcal{A}_d}_{\textit{Schur}}$$

Motivation

- **Robust stability (synthesis):**
  - Find a matrix L of appropriate dimensions

$$\underbrace{e^{Ah} + \left(\int_0^h e^{A\tau} B d\tau\right) L}_{\textit{Schur}} \;\Longleftarrow\; \forall (A, B) \in \mathcal{A}_c \times \mathcal{B}_c$$

  - Nonlinear and nonconvex parameter dependence
  - Hard to solve
  - Robust performance similar difficulty!

Motivation

- **Quadratic stability:**
  - Continuous-time polytopic system

$$A'S + SA < 0,\ \ S > 0,\ \ \forall A \in \mathcal{A}_c$$

$$\Downarrow \ \forall h > 0$$

  - Sampled-data polytopic system

$$\left\{ \begin{array}{rcl} \dot{P}(t) + A'P(t) + P(t)A & < & 0 \\ P(0) = P(h) = S & > & 0 \end{array} \right. \quad t \in [0, h),\ \ \forall A \in \mathcal{A}_c$$

$$\Updownarrow$$

$$e^{A'h} S e^{Ah} - S < 0,\ \ S > 0,\ \ \forall A \in \mathcal{A}_c$$

Motivation

- **Differential Linear Matrix Inequality (DLMI):** General form arising in $\mathcal{H}_\infty$ control design

$$\begin{bmatrix} \dot{P}(t) + F'P(t) + P(t)F & P(t)J & G' \\ \bullet & -\gamma^2 I & 0 \\ \bullet & \bullet & -I \end{bmatrix} < 0, \;\; t \in [0, h)$$

$$P(0), \;\; P(h) \;\; \Longleftarrow \;\; LMI$$

  - Linearity with respect to the parameters
  - Feasible solution $P(t)$ of specific form:

    piecewise linear (Allerhand & Shaked, 2013) and (Briat, 2013)
    polynomial, $sinc(\cdot)$ functions, ...

Motivation

- **Sampled-data control:** Zero order hold

$$u \in \mathcal{U} \iff u(t) = \underbrace{u[k]}_{u(t_k)}, \ \forall t \in [t_k, t_{k+1}), \ \forall k \in \mathbb{N}$$

  - Sampling times $t_0 = 0, \quad t_{k+1} - t_k = h > 0, \ \forall k \in \mathbb{N}$
  - $\mathcal{H}_2$ and $\mathcal{H}_\infty$ optimization subject to $u \in \mathcal{U}$
  - Three seminal books:

    J. R. Ragazzini & G. F. Franklin, 1958
    T. Chen & B. A. Francis, 1995
    A. Ichicawa & H. Katayama, 2001

Sampled-data control

- LTI plant

$$\dot{x}(t) = Ax(t) + Bu(t) + E_c w_c(t),\ x(0) = x_0$$
$$z(t) = C_c x(t) + D_c u(t)$$

  - $x(\cdot) \in \mathbb{R}^{n_x}$ is the state
  - $w_c(\cdot) \in \mathbb{R}^{r_c}$ is the exogenous continuous-time input
  - $z(\cdot) \in \mathbb{R}^{n_z}$ is the controlled output
  - $u(\cdot) \in \mathbb{R}^{n_u}$ is the control input

$$u \in \mathcal{U} \iff u(t) = u[k],\ t \in [t_k, t_{k+1}),\ \forall k \in \mathbb{N}$$

Sampled-data control

- State feedback with control noise

$$u[k] = \hat{C}x[k] + E_d \underbrace{w_d[k-1]}_{\in \mathbb{R}^{r_d}}$$

- Dynamic output feedback with measurement noise

$$\underbrace{y[k]}_{\in \mathbb{R}^{n_y}} = C_d x[k] + E_d w_d[k-1]$$

$$\hat{x}[k] = \hat{A}\hat{x}[k-1] + \hat{B}y[k]$$
$$u[k] = \hat{C}\hat{x}[k-1] + \hat{D}y[k]$$

  - Initial condition $\hat{x}[-1] = 0$
  - Full order controller $\hat{x}[\cdot] \in \mathbb{R}^{n_c}$ with $n_c = n_u + n_x$!

Sampled-data control

- **Sampled-data control design in a general framework**

$$\Downarrow$$

Hybrid Systems

$$+$$

Sampled-data control

- **Sampled-data control design in a general framework**

$$\Downarrow$$

Hybrid Systems

$$+$$

Bellman's Principle of Optimality

Hybrid systems

- The closed-loop sampled-data system can be rewritten as a

$$\begin{aligned} \dot{\psi}(t) &= F\psi(t) + J_c w_c(t) \\ z(t) &= G\psi(t) \\ \psi(t_k) &= H\psi(t_k^-) + J_d w_d[k-1] \end{aligned}$$

valid in the time interval $t \in [t_k, t_{k+1})$ for all $k \in \mathbb{N}$

- Necessary and sufficient condition for asymptotic stability

$$\psi(t_k) \to \psi(t_{k+1}^-) \to \psi(t_{k+1}), \ \forall k \in \mathbb{N}$$

- Performance index calculation

Hybrid systems

- State space matrices

$$\underbrace{F,\ J_c,\ G}_{open-loop\ \ system\ \ data}$$

and

$$\underbrace{H,\ J_d}_{controller\ \ state\ \ space\ \ matrices}$$

- The process starts at $t_0 = 0$ with

$$\psi(0) = H\underbrace{\psi(0^-)}_{\psi_0} + J_d w_d[-1]$$

Hybrid systems

- Given $\gamma \in \mathbb{R}_+$, consider the function

$$\rho_\gamma(\psi(0)) = \sup_{w_c \in \mathcal{L}_2,\ w_d \in \ell_2} \|z\|_2^2 - \gamma^2 \left(\underbrace{\|w_c\|_2^2 + \|w_d\|_2^2}_{\textit{disturbance}} \right)$$

  - $\mathcal{H}_\infty$ **sampled-data control:**

$$J_\infty = \inf \left\{ \gamma^2 \ : \ \rho_\gamma(0) \leq 0 \right\}$$

  - $\mathcal{H}_2$ **sampled-data control:**

$$J_2 = \sum_{\ell=1}^{r_c + r_d} \rho_\infty(\psi_\ell(0))$$

$\mathcal{H}_2$ performance

- **Proposition:** Let $h \in \mathbb{R}_+$ be given. The hybrid linear system is asymptotically stable and the $\mathcal{H}_2$ performance index equals the optimal solution to the convex programming problem

$$J_2 = \inf_{P(\cdot)} \left\{ \mathbf{tr}(J_c' P(h) J_c) + \mathbf{tr}(\underbrace{J_d' P(0) J_d}_{controller}) \right\}$$

subject to the **DLMI**

$$\dot{P}(t) + F' P(t) + P(t) F + G' G < 0$$

satisfying the boundary condition

$$\underbrace{\begin{bmatrix} P(h) & H' \\ \bullet & P(0)^{-1} \end{bmatrix}}_{controller} > 0$$

$\mathcal{H}_2$ performance

- Matrix $S = P(0) > 0$ satisfies the strict Lyapunov inequality

$$e^{F'h}H'SHe^{Fh} < S - \underbrace{\int_0^h e^{F't}G'Ge^{Ft}dt}_{\geq 0}$$

which admits a solution if and only if He^{Fh} is Schur stable

- The optimal sampled-data $\mathcal{H}_2$ control follows from

$$\inf_{controller, P(\cdot)} \left\{ \mathbf{tr}(J_c' P(h) J_c) + \mathbf{tr}(\underbrace{J_d' P(0) J_d}_{controller}) \right\} \quad \rightarrow \quad CONVEX$$

$\mathcal{H}_\infty$ performance

- **Proposition:** Let $h \in \mathbb{R}_+$ be given. The hybrid linear system is asymptotically stable and the $\mathcal{H}_\infty$ performance index equals the optimal solution to the convex programming problem

$$J_\infty = \inf_{P(\cdot),\gamma} \gamma^2$$

subject to the **DLMI**

$$\dot{P}(t) + F'P(t) + P(t)F + \gamma^{-2}P(t)J_cJ_c'P(t) + G'G < 0$$

satisfying the boundary condition

$$\underbrace{\begin{bmatrix} P(h) & H' & 0 \\ \bullet & P(0)^{-1} & J_d \\ \bullet & \bullet & \gamma^2 I \end{bmatrix}}_{\textit{controller}} > 0$$

$\mathcal{H}_\infty$ performance

- A solution exists provided that $\gamma > 0$ is large enough
- The optimal sampled-data $\mathcal{H}_\infty$ control follows from

$$J_\infty = \inf_{\textit{controller}, P(\cdot), \gamma} \left\{ \gamma^2 \; : \; \underbrace{\begin{bmatrix} P(h) & H' & 0 \\ \bullet & P(0)^{-1} & J_d \\ \bullet & \bullet & \gamma^2 I_{r_d} \end{bmatrix}}_{\textit{controller}} > 0 \right\}$$

$$\Downarrow$$

CONVEX

Boundary conditions

- **Aperiodic sampling**

$$\underbrace{\begin{bmatrix} P(h) & H' & 0 \\ \bullet & P(0)^{-1} & J_d \\ \bullet & \bullet & \gamma^2 I \end{bmatrix}}_{controller} > 0, \; h \in [h_{min}, h_{max}]$$

guaranteed performance (upper bound) optimization!

- **LTI systems**

$$H = I, \; J_d = 0 \implies \underbrace{P(0) = P(h) = P(t) > 0}_{stationary \;\; solution}, \; \forall t \in [0, h)$$

Linearization

- Four square blocks partitioning, (Scherer, 1995)

$$P(t) = \begin{bmatrix} X(t) & V(t) \\ \bullet & \hat{X}(t) \end{bmatrix}, \; P(t)^{-1} = \begin{bmatrix} Y(t) & U(t) \\ \bullet & \hat{Y}(t) \end{bmatrix}$$

  leads to:
  - **Differential LMIs** and boundary **LMI** constraints
  - One-to-one change of variables yields the optimal controller
  - Full order controller $n_c = n_x$ due to pole / zero cancelation!

  - Generalization to **Markov jump linear systems**

Numerical issue

- **Class of convex problems to be solved**

$$f^* = \inf_{(P_0,P_h)\in\Omega} \left\{ f(P_0,P_h) \ : \ \mathcal{L}(\dot{P}(t),P(t)) < 0,\ t\in[0,h) \right\}$$

  - $f(\cdot,\cdot)$ is linear
  - $\mathcal{L}(\cdot,\cdot)$ is linear
  - $(P(0),P(h)) = (P_0,P_h)\in\Omega$ is expressed by LMIs

$$\Uparrow$$

$$P(t) = \sum_{i=0}^{n_\phi-1} X_i\phi_i(t)$$

Numerical issue

- **Piecewise linear:** For $\eta = h/n_\phi$ and $\phi_i(t) = \phi(t - i\eta)$

$$\phi(t) = \begin{cases} 1 - \frac{|t|}{\eta} & , \quad |t| \leq \eta \\ 0 & , \quad \text{otherwise} \end{cases}$$

- $P(t)$ is continuous and is feasible if and only if

$$\begin{aligned} \mathcal{L}\left(\frac{X_{i+1} - X_i}{\eta}, X_i\right) &< 0 \\ \mathcal{L}\left(\frac{X_{i+1} - X_i}{\eta}, X_{i+1}\right) &< 0 \\ (X_0, X_{n_\phi}) \in \Omega \end{aligned}$$

Optimal solution in only one shot $\Leftarrow$ $2n_\phi + 1$ LMIs!

Numerical issue

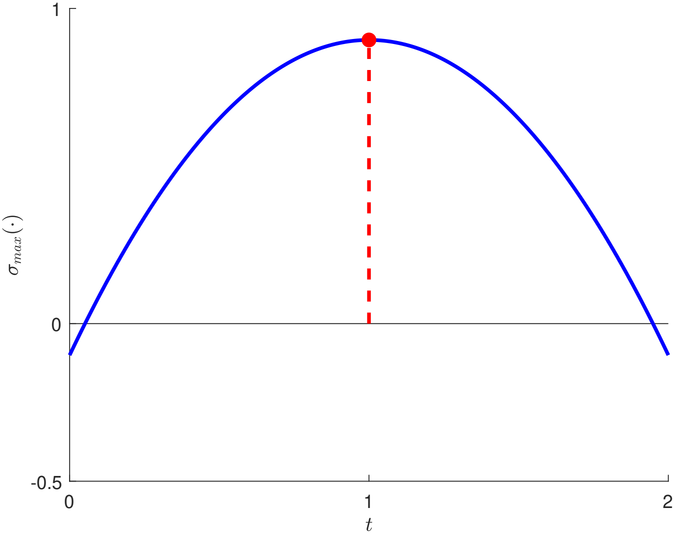
- **Polynomial:**

$$\phi_i(t) = \left(\frac{t}{h}\right)^i$$

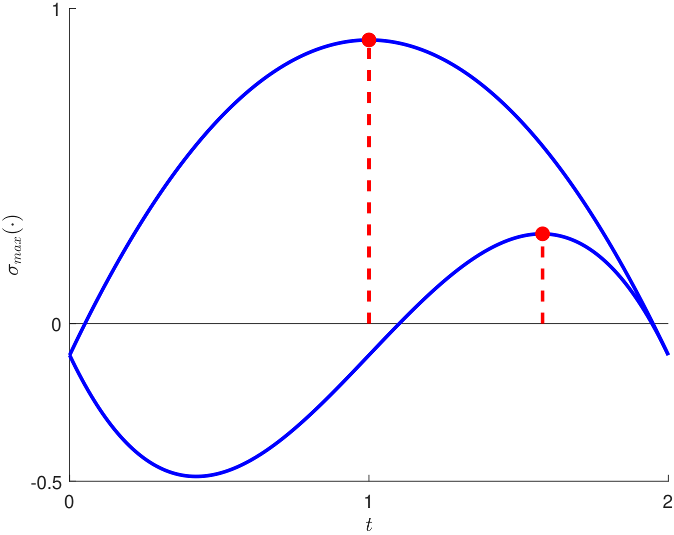
- Iterative solution by **outer linearization**
  - Simple to solve convex (LMI) subproblem
  - Global convergence
  - Optimality test

$$\max_{t\in[0,h]} \sigma_{max}\left(\mathcal{L}\left(\dot{P}(t), P(t)\right)\right) < 0$$

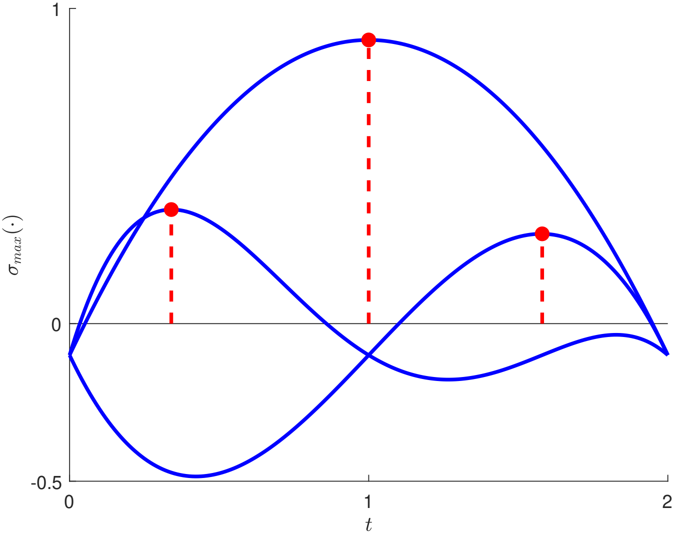
Outer linearization



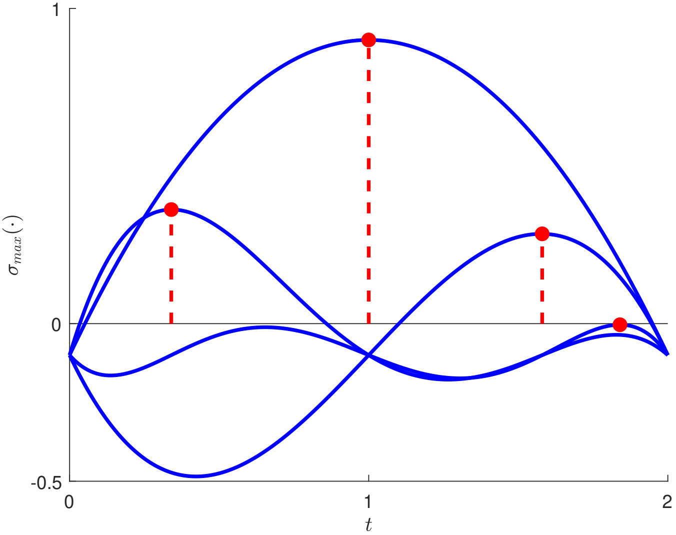
Outer linearization



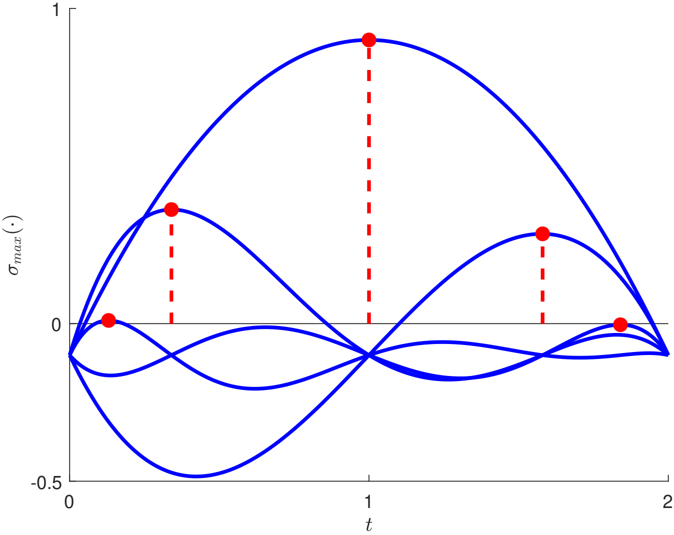
Outer linearization



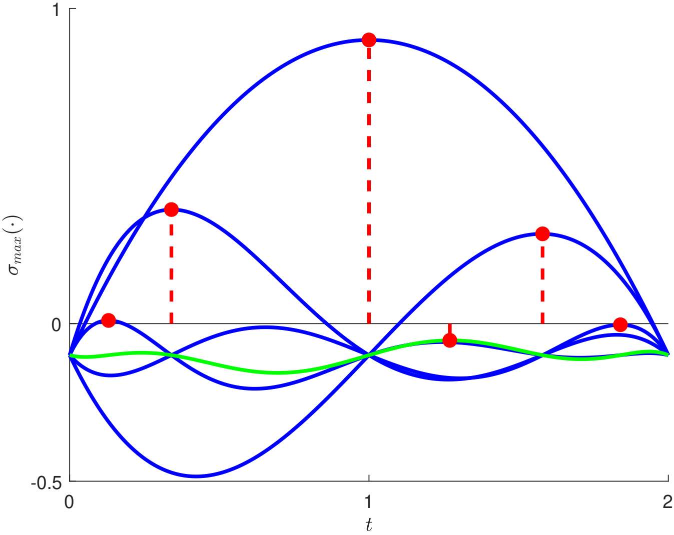
Outer linearization



Outer linearization



Outer linearization



Example

- Example borrowed from (**Ichicawa & Katayama, 2001**)

$$A = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}, \; B = E_c = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \; C_d' = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

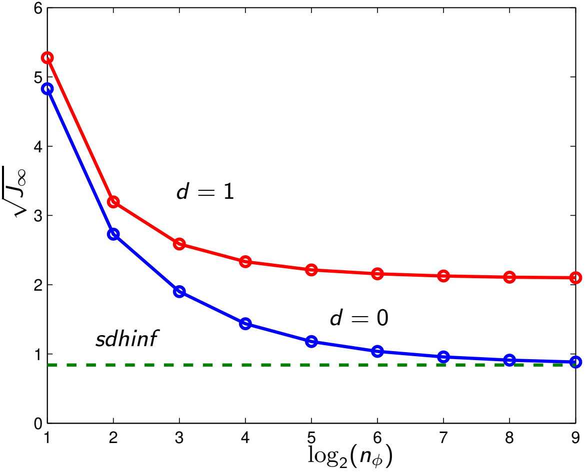
$$C_c = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \; D_c = \begin{bmatrix} 0 \\ d \end{bmatrix}, \; E_d = d, \quad h = 1.0$$

For $d = 1$, the optimal $\mathcal{H}_\infty$ sampled-data controller

$$\mathcal{C}^*(\zeta) = \frac{-0.3225\zeta^3 + 0.8686\zeta^2 - 8.012\ 10^{-10}\zeta}{\zeta^3 + 0.2865\zeta^2 + 0.01608\zeta + 1.697\ 10^{-12}}$$

imposes the performance cost $\gamma_* = 2.16$.

Example - Piecewise linear



Comparison

- **Piecewise linear**
  - Non iterative method
  - $n_\phi \approx 2^6$ time segments
- **Outer linearization**
  - Iterative method
  - Polynomial function of degree $n_\phi \approx 2 \times 6$

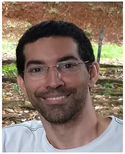
Conclusion

- **Open problems:**
  - **Dynamic output feedback control:** in the context of Markov Jump Linear Systems.
  - **Nonuniform sampling:** optimality

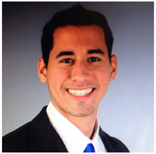
Conclusion

- This research has been developed in collaboration with

Gabriela Werner Gabriel



Rafael Fernandes Cunha



Tiago Rocha Gonçalves

Conclusion

Thank you for your attention!!!